

All order ε -expansion of Gauss hypergeometric functions with integer and half/integer values of parameters*

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ABSTRACT: It is proved that the Laurent expansion of the following Gauss hypergeometric functions,

$$\begin{aligned} & {}_2F_1(I_1 + a\varepsilon, I_2 + b\varepsilon; I_3 + c\varepsilon; z), \\ & {}_2F_1(I_1 + a\varepsilon, I_2 + b\varepsilon; I_3 + \frac{1}{2} + c\varepsilon; z), \\ & {}_2F_1(I_1 + \frac{1}{2} + a\varepsilon, I_2 + b\varepsilon; I_3 + c\varepsilon; z), \\ & {}_2F_1(I_1 + \frac{1}{2} + a\varepsilon, I_2 + b\varepsilon; I_3 + \frac{1}{2} + c\varepsilon; z), \\ & {}_2F_1(I_1 + \frac{1}{2} + a\varepsilon, I_2 + \frac{1}{2} + b\varepsilon; I_3 + \frac{1}{2} + c\varepsilon; z), \end{aligned}$$

where I_1, I_2, I_3 are an arbitrary integer nonnegative numbers, a, b, c are an arbitrary numbers and ε is an arbitrary small parameters, are expressible in terms of the harmonic polylogarithms of Remiddi and Vermaseren with polynomial coefficients. An efficient algorithm for the calculation of the higher-order coefficients of Laurent expansion is constructed. Some particular cases of Gauss hypergeometric functions are also discussed.

KEYWORDS: Differential and Algebraic Geometry, NLO Computations.

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1. Introduction

One of the most powerful techniques for calculating Feynman diagrams is based on their presentation in terms of hypergeometric functions. We will call this the hypergeometric function representation of Feynman diagrams. Such a representation can be used for numerical evaluation, construction of the asymptotic expansion, *etc.* One of the unsolved problems in this program is obtaining the proper representation for a diagram with an arbitrary number of legs and loops. Direct use of α - or Feynman parameters representations [1] is not very helpful in solving this problem. The Mellin-Barnes technique is restricted to several topologies [2–5]. The negative dimension approach [6] has a similar restriction [7]. The most investigated diagrams are the master integrals (typically, integrals with the power of each propagator equal to unity). The differential [8] and/or difference equation [9] techniques are usually used to obtain such representations. The known cases include the one-loop diagrams [10, 11], two-loop propagator-type diagrams with special mass and momentum values [12], several three-loop bubble-type diagrams [3], three-loop vertex-type diagrams [13], and four-loop bubble-type diagrams [14]. For practical application however, it is necessary to construct the ε -expansion (Laurent expansion) of hypergeometric functions. There is some evidence that the multiple polylogarithms [15–17],

$$\text{Li}_{k_1, k_2, \dots, k_n}(z_1, z_2, \dots, z_n) = \sum_{m_1 > m_2 > \dots > m_n > 0} \frac{z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}}{m_1^{k_1} m_2^{k_2} \dots m_n^{k_n}}. \quad (1.1)$$

are sufficient for parametrizing the coefficients of the ε -expansion of some, but not all,¹ hypergeometric functions [18].

In some particular cases, the result of the Laurent expansion can be written in terms of simpler functions. In particular, at the present moment, it is commonly accepted [19] that the generalized hypergeometric functions with an arbitrary set of integer parameters can be presented in terms of harmonic polylogarithms [20]. The idea of the proof is based on the properties of *nested sums* [21]: the analytical coefficients of the ε -expansion of any generalized hypergeometric function with integer parameters can be reduced to a set of some basic *harmonic series* of the type

$$\sum_{j=1}^{\infty} \frac{z^j}{j^c} S_{a_1}(j-1) \cdots S_{a_p}(j-1), \tag{1.2}$$

where z is an arbitrary argument and $S_a(j)$ is an harmonic sum defined as $S_a(j) = \sum_{k=1}^j \frac{1}{k^a}$. Series of this type are expressible in terms of the Remiddi-Vermaseren harmonic polylogarithms.

However for hypergeometric functions with half-integer values of parameters, the new type of sums, *multiple (inverse) binomial sums* [22, 4, 5, 23] are generated:

$$\Sigma_{a_1, \dots, a_p; b_1, \dots, b_q; c}^{(k)}(z) \equiv \sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}^k} \frac{z^j}{j^c} S_{a_1}(j-1) \cdots S_{a_p}(j-1) S_{b_1}(2j-1) \cdots S_{b_q}(2j-1). \tag{1.3}$$

For particular values of k , the sums (1.3) are called

$$k = \left\{ \begin{array}{l} 0 \text{ generalized harmonic} \\ 1 \text{ inverse binomial} \\ -1 \text{ binomial} \end{array} \right\} \text{ sums .}$$

At the present moment, there is no proof that any *multiple (inverse) binomial sums* can be expressed in terms of harmonic polylogarithms only. This problem was investigated in ref. [23] for *multiple inverse binomial sums* up to *weight 4*. In ref. [24], it was shown that some of the *multiple inverse binomial sums* are not expressible in terms of harmonic polylogarithms of simple argument. In ref. [25], the results of ref. [23] were extended on the case of special combinations of *multiple binomial sums* and *multiple generalized harmonic sums*. However, the Laurent expansion of a hypergeometric function in general contains combinations of *multiple sums*. These combinations may be expressed in terms of harmonic polylogarithms. From this point of view, the construction of the analytical coefficients of the ε -expansion of hypergeometric functions can be done independently from existing results for each individual *multiple sum*.²

¹We are thankful to S. Weinzierl for this information.

²We are indebted to A. Davydychev for discussion on this subject.

The simplest hypergeometric function is the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$, [26–28]. It satisfies the second-order differential equation

$$\frac{d}{dz} \left(z \frac{d}{dz} + c - 1 \right) w(z) = \left(z \frac{d}{dz} + a \right) \left(z \frac{d}{dz} + b \right) w(z), \quad (1.4)$$

and admits the series representation

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad (1.5)$$

where $(a)_k = \Gamma(a + k)/\Gamma(a)$ is the Pochhammer symbol.

The primary aim of this paper is to prove the following:

Theorem 1. *The all-order ε -expansions of the Gauss hypergeometric functions*

$${}_2F_1(I_1 + a\varepsilon, I_2 + b\varepsilon; I_3 + c\varepsilon; z), \quad (1.6a)$$

$${}_2F_1(I_1 + a\varepsilon, I_2 + b\varepsilon; I_3 + \frac{1}{2} + c\varepsilon; z), \quad (1.6b)$$

$${}_2F_1(I_1 + \frac{1}{2} + a\varepsilon, I_2 + b\varepsilon; I_3 + c\varepsilon; z), \quad (1.6c)$$

$${}_2F_1(I_1 + \frac{1}{2} + a\varepsilon, I_2 + b\varepsilon; I_3 + \frac{1}{2} + c\varepsilon; z), \quad (1.6d)$$

$${}_2F_1(I_1 + \frac{1}{2} + a\varepsilon, I_2 + \frac{1}{2} + b\varepsilon; I_3 + \frac{1}{2} + c\varepsilon; z), \quad (1.6e)$$

where $\{I_k\}$ are integer numbers, a, b, c are an arbitrary numbers, and ε is an arbitrary small parameter, are expressible in terms of Remiddi-Vermaseren harmonic polylogarithms with rational coefficients.

2. All-order ε -expansion

2.1 Non-zero values of the ε -dependent part

It is well known that any Gauss hypergeometric function may be expressed as a linear combination of two other hypergeometric functions with parameters differing from the original ones by an integer [26–29, 25]. Such a representation will be called a *reduction*, and the explicit algorithm will be called a *reduction algorithm*. Using the algorithm described in ref. [25], the result of the reduction can be written as

$$P(a, b, c, z) {}_2F_1(a + I_1, b + I_2; c + I_3; z) = \left\{ Q_1(a, b, c, z) \frac{d}{dz} + Q_2(a, b, c, z) \right\} {}_2F_1(a, b; c; z), \quad (2.1)$$

where a, b, c , are any fixed numbers, P, Q_1, Q_2 are polynomial in parameters a, b, c and argument z , and I_1, I_2, I_3 any integer numbers.

All the hypergeometric functions (1.6) listed in Theorem 1 can be reduced to functions with I_1, I_2, I_3 equal to zero for half-integer values of parameters, and to unity for integer

ones. In this way, all the hypergeometric functions of Theorem 1 are expressible in terms of the following five basic functions and their first derivatives:

$$\begin{aligned}
 & {}_2F_1(a_1\varepsilon, a_2\varepsilon; 1 + c\varepsilon; z), \quad {}_2F_1(a_1\varepsilon, a_2\varepsilon; \frac{1}{2} + f\varepsilon; z), \quad (2.2a) \\
 & {}_2F_1(\frac{1}{2} + b\varepsilon, a\varepsilon; 1 + c\varepsilon; z), \quad {}_2F_1(\frac{1}{2} + b\varepsilon, a\varepsilon; \frac{1}{2} + f\varepsilon; z), \quad {}_2F_1(\frac{1}{2} + b_1\varepsilon, \frac{1}{2} + b_2\varepsilon; \frac{1}{2} + f\varepsilon; z). \quad (2.2b)
 \end{aligned}$$

It was shown in ref. [25] that only the two hypergeometric functions (and their first derivative) of type (2.2a) are algebraically independent. The other three, (2.2b), are algebraically expressible in terms of ${}_2F_1(a_1\varepsilon, a_2\varepsilon; \frac{1}{2} + f\varepsilon; z)$. Consequently, in order to prove Theorem 1, it is sufficient to show that the analytical coefficients of the ε -expansion of the first two hypergeometric functions (2.2a) are expressible in terms of Remiddi-Vermaseren polylogarithms.

2.1.1 Integer values of ε -independent parameters

Let us start from expansion of Gauss hypergeometric functions with integer values of parameters, and consider the function ${}_2F_1(a_1\varepsilon, a_2\varepsilon; 1 + c\varepsilon; z)$. In ref. [30], the all-order ε -expansions for this function and its first derivative were constructed in terms of multiple polylogarithms of one variable [16, 17]. These multiple polylogarithms may be expressed as iterated integrals³ and have the expansion

$$\text{Li}_{k_1, k_2, \dots, k_n}(z) = \sum_{m_1 > m_2 > \dots > m_n > 0} \frac{z^{m_1}}{m_1^{k_1} m_2^{k_2} \dots m_n^{k_n}}. \quad (2.6)$$

Similar results were also derived (without explicit form of coefficients) in ref. [19] via nested sums approach. We will follow the idea of ref. [30].

The Gauss hypergeometric function ${}_2F_1(a_1\varepsilon, a_2\varepsilon; 1 + c\varepsilon; z)$ is the solution of the differential equation

$$\frac{d}{dz} \left(z \frac{d}{dz} + c\varepsilon \right) w(z) = \left(z \frac{d}{dz} + a_1\varepsilon \right) \left(z \frac{d}{dz} + a_2\varepsilon \right) w(z), \quad (2.7)$$

³Recall that multiple polylogarithms can be expressed as iterated integrals of the form

$$\text{Li}_{k_1, \dots, k_n}(z) = \int_0^z \underbrace{\frac{dt}{t} \circ \frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{k_1 - 1 \text{ times}} \circ \frac{dt}{1-t} \circ \dots \circ \underbrace{\frac{dt}{t} \circ \frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{k_n - 1 \text{ times}} \circ \frac{dt}{1-t}, \quad (2.3)$$

where, by definition

$$\int_0^z \underbrace{\frac{dt}{t} \circ \frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{k_1 - 1 \text{ times}} \circ \frac{dt}{1-t} = \int_0^z \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \dots \int_0^{t_{k_1-2}} \frac{dt_{k_1-1}}{t_{k_1-1}} \int_0^{t_{k_1-1}} \frac{dt_{k_1}}{1-t_{k_1}}. \quad (2.4)$$

The integral (2.3) is an iterated Chen integral [31] (see also [32]) w.r.t. the two differential forms $\omega_0 = dz/z$ and $\omega_1 = \frac{dz}{1-z}$, so that

$$\text{Li}_{k_1, \dots, k_n}(z) = \int_0^z \omega_0^{k_1-1} \omega_1 \dots \omega_0^{k_n-1} \omega_1. \quad (2.5)$$

with boundary conditions $w(0) = 1$ and $z \frac{d}{dz} w(z) \Big|_{z=0} = 0$. Eq. (2.7) is valid in each order of ε , so that in terms of coefficients functions $w_k(z)$ defined as

$$w(z) = \sum_{k=0}^{\infty} w_k(z) \varepsilon^k, \tag{2.8}$$

it can be written

$$(1-z) \frac{d}{dz} \left(z \frac{d}{dz} \right) w_k(z) = \left(a_1 + a_2 - \frac{c}{z} \right) \left(z \frac{d}{dz} \right) w_{k-1}(z) + a_1 a_2 w_{k-2}(z) \tag{2.9}$$

for $k \geq 0$ with

$$w_0(z) = 1, \tag{2.10a}$$

$$w_k(z) = 0, \quad k < 0. \tag{2.10b}$$

The boundary conditions for the coefficient functions are

$$w_k(0) = 0, \quad k \geq 1, \tag{2.11a}$$

$$z \frac{d}{dz} w_k(z) \Big|_{z=0} = 0, \quad k \geq 0. \tag{2.11b}$$

Let us introduce a new function $\rho(z)$ defined by⁴

$$\rho(z) = z \frac{d}{dz} w(z) = \sum_{k=0}^{\infty} \rho_k(z) \varepsilon^k, \tag{2.12}$$

where the coefficient functions satisfy

$$\rho_k(z) = z \frac{d}{dz} w_k(z). \tag{2.13}$$

The boundary conditions for these new functions follow from eq. (2.11):

$$\rho_k(0) = 0, \quad k \geq 0. \tag{2.14}$$

Eq. (2.9) can be rewritten as a system of two first-order differential equations:

$$\begin{aligned} (1-z) \frac{d}{dz} \rho_i(z) &= \left(a_1 + a_2 - \frac{c}{z} \right) \rho_{i-1}(z) + a_1 a_2 w_{i-2}(z), \\ z \frac{d}{dz} w_i(z) &= \rho_i(z). \end{aligned} \tag{2.15}$$

The solution of this system can be presented in an iterated form:

$$\begin{aligned} \rho_i(z) &= (a_1 + a_2 - c) \int_0^z \frac{dt}{1-t} \rho_{i-1}(t) + a_1 a_2 \int_0^z \frac{dt}{1-t} w_{i-2}(t) - c w_{i-1}(z), \quad i \geq 1, \\ w_i(z) &= \int_0^z \frac{dt}{t} \rho_i(t), \quad i \geq 1. \end{aligned} \tag{2.16}$$

⁴We may note that

$${}_2F_1 \left(\begin{matrix} 1 + a_1 \varepsilon, 1 + a_2 \varepsilon \\ 2 + c \varepsilon \end{matrix} \middle| z \right) = \frac{1 + c \varepsilon}{z} \sum_{k=0}^{\infty} \left[\frac{\rho_{k+2}(z)}{a_1 a_2} \right] \varepsilon^k.$$

Taking into account that $w_0(z) = 1$ and $\rho_0(z) = 0$ (the ε -expansion of $\rho(z)$ begins with the term linear in ε), we obtain the first few coefficients,

$$\rho_1(z) = w_1(z) = 0, \tag{2.17a}$$

$$\frac{\rho_2(z)}{a_1 a_2} = -\ln(1-z) \equiv H(1; z), \tag{2.17b}$$

$$\frac{w_2(z)}{a_1 a_2} = \text{Li}_2(z) \equiv H(0, 1; z), \tag{2.17c}$$

$$\frac{\rho_3(z)}{a_1 a_2} = \gamma_c \frac{1}{2} \ln^2(1-z) - c \text{Li}_2(z) \equiv \gamma_c H(1, 1; z) - c H(0, 1; z), \tag{2.17d}$$

$$\frac{w_3(z)}{a_1 a_2} = \gamma_c S_{1,2}(z) - c \text{Li}_3(z) \equiv \gamma_c H(0, 1, 1; z) - c H(0, 0, 1; z), \tag{2.17e}$$

where we have defined $\gamma_c = a_1 + a_2 - c$, and $\text{Li}_n(z)$ and $S_{a,b}(z)$ are the classical and Nielsen polylogarithms [33, 34], respectively:

$$S_{a,b}(z) = \frac{(-1)^{a+b-1}}{(a-1)! b!} \int_0^1 d\xi \frac{\ln^{a-1} \xi \ln^b(1-z\xi)}{\xi}, \quad S_{a,1}(z) = \text{Li}_{a+1}(z).$$

The functions $H(\vec{A}; z)$ are the Remiddi-Vermaseren harmonic polylogarithms [20], and \vec{A} is a multiple index including only entries 0 and 1.

From the representation (2.16) and result for the first few coefficients (2.17) we may derive the following observations:

Corollary 1. *The all-order ε -expansion of the function ${}_2F_1(a_1\varepsilon, a_2\varepsilon; 1+c\varepsilon; z)$ may be written in terms of harmonic polylogarithms $H_{\vec{A}}(z)$ only, where the multiple index \vec{A} includes only the values 0 and 1.*

Corollary 2. *The analytical coefficient of ε^k in the expansion of ${}_2F_1(a_1\varepsilon, a_2\varepsilon; 1+c\varepsilon; z)$ includes only functions of weight k with numerical coefficients.*

Corollary 3. *The non-constant terms of the ε -expansion of ${}_2F_1(a_1\varepsilon, a_2\varepsilon; 1+c\varepsilon; z)$ are proportional to the product $a_1 a_2$ in any order of ε .*

The first and the last statement follows from the representation⁵ (2.16), the explicit value of coefficients functions $w_k(z)$, $k = 0, 1, 2$ (eqs. (2.17)), and definition of harmonic polylogarithms [20]. The second statement follows from the form of the solution of eq. (2.16).

The relation between harmonic polylogarithms $H_{\vec{A}}(z)$, with multiple index \vec{A} including only 0 and 1, and multiple polylogarithms of one variable (eq. 2.6) is well known [21] and follows from the proper definition (see section 2 in ref. [20]):

$$\text{Li}_{k_1, k_2, \dots, k_n}(z) = H(\underbrace{0, 0, \dots, 0}_{k_1-1 \text{ times}}, \underbrace{1, 0, 0, \dots, 0}_{k_2-1 \text{ times}}, \dots, \underbrace{0, 0, \dots, 0}_{k_n-1 \text{ times}}, 1; z). \tag{2.18}$$

⁵The Corollary 3 follows also from general properties of hypergeometric functions.

By continued iterations of eq. (2.16) and eq. (2.17), we have reproduced all coefficients of the ε -expansion of the Gauss hypergeometric function presented in eq. (4.7) of [25]. For the coefficients functions $\rho_4(z)$, $\omega_4(z)$, and $\rho_5(z)$, we find a more compact form. We also obtain the higher-order terms $\omega_5(z)$ and $\rho_6(z)$ of the ε -expansion. The results are⁶

$$\frac{\rho_4(z)}{a_1 a_2} = -\frac{1}{6}\gamma_c^2 \ln^3(1-z) + (c\gamma_c - a_1 a_2) \ln(1-z) \text{Li}_2(z) + c^2 \text{Li}_3(z) + (c\gamma_c - 2a_1 a_2) \text{S}_{1,2}(z) ,$$

$$\frac{\omega_4(z)}{a_1 a_2} = c^2 \text{Li}_4(z) - \frac{1}{2}(c\gamma_c - a_1 a_2) [\text{Li}_2(z)]^2 + \gamma_c^2 \text{S}_{1,3}(z) + (c\gamma_c - 2a_1 a_2) \text{S}_{2,2}(z) , \quad (2.19)$$

$$\begin{aligned} \frac{\rho_5(z)}{a_1 a_2} = & \frac{1}{24}\gamma_c^3 \ln^4(1-z) - c^3 \text{Li}_4(z) - c\gamma_c^2 \text{S}_{1,3}(z) \\ & - c(c\gamma_c - a_1 a_2) \ln(1-z) \text{Li}_3(z) - c(c\gamma_c - 2a_1 a_2) \text{S}_{2,2}(z) \\ & - \gamma_c(c\gamma_c - a_1 a_2) \ln(1-z) \left[\frac{1}{2} \ln(1-z) \text{Li}_2(z) + \text{S}_{1,2}(z) \right] , \end{aligned} \quad (2.20)$$

$$\begin{aligned} \frac{\omega_5(z)}{a_1 a_2} = & \gamma_c^3 \text{S}_{1,4}(z) - c^3 \text{Li}_5(z) - c\gamma_c^2 \text{S}_{2,3}(z) - c(c\gamma_c - 2a_1 a_2) \text{S}_{3,2}(z) \\ & + (c\gamma_c - a_1 a_2) \left[\gamma_c \text{Li}_2(z) \text{S}_{1,2}(z) - \gamma_c F_1(z) - c F_2(z) \right] , \end{aligned} \quad (2.21)$$

$$\begin{aligned} \frac{\rho_6(z)}{a_1 a_2} = & -\gamma_c^4 \frac{1}{120} \ln^5(1-z) + c^4 \text{Li}_5(z) + c^2 \gamma_c^2 \text{S}_{2,3}(z) \\ & + \frac{1}{6}\gamma_c^2 (c\gamma_c - a_1 a_2) \left[\ln^3(1-z) \text{Li}_2(z) + 3 \ln^2(1-z) \text{S}_{1,2}(z) + 6 \ln(1-z) \text{S}_{1,3}(z) \right] \\ & + \frac{1}{2}c(c\gamma_c - a_1 a_2) \ln(1-z) \left[\gamma_c \ln(1-z) \text{Li}_3(z) + 2c \text{Li}_4(z) \right] \\ & - (c - a_1)(c - a_2)(c\gamma_c - 2a_1 a_2) \ln(1-z) \text{S}_{2,2}(z) \\ & + a_1 a_2 (c\gamma_c - a_1 a_2) \left[\frac{1}{2} \ln(1-z) [\text{Li}_2(z)]^2 - 2 \text{Li}_2(z) \text{S}_{1,2}(z) + 2 F_1(z) \right] \\ & + (c\gamma_c - 2a_1 a_2) \left[\gamma_c^2 \text{S}_{1,4}(z) + c^2 \text{S}_{3,2}(z) \right] , \end{aligned} \quad (2.22)$$

where we have introduced two new functions:

$$F_1(z) = \int_0^z \frac{dx}{x} \ln^2(1-x) \text{Li}_2(x) , \quad (2.23)$$

$$F_2(z) = \int_0^z \frac{dx}{x} \ln(1-x) \text{Li}_3(x) . \quad (2.24)$$

⁶The FORM [45] representation of these expressions can be extracted from ref. [48].

There is an algebraic relation⁷ between these two functions:

$$\begin{aligned}
 F_2(1-z) &= F_1(z) - 2 \ln z S_{1,3}(z) + 2S_{2,3}(z) - \text{Li}_2(z) S_{1,2}(z) - \ln z \ln(1-z) S_{1,2}(z) \\
 &\quad - \frac{1}{6} \ln^3(1-z) \ln^2 z - \frac{1}{2} \ln z \ln^2(1-z) \text{Li}_2(z) + \frac{1}{2} \zeta_2 \ln^2(1-z) \ln z \\
 &\quad - \zeta_2 S_{1,2}(z) - \zeta_3 \text{Li}_2(1-z) - \zeta_5,
 \end{aligned} \tag{2.25}$$

where

$$F_1(1) = 2\zeta_3\zeta_2 - \zeta_5 \sim 2.9176809 \dots$$

In this way, at the order of weight 5, one new function,⁸ F_1 , which is not expressible in terms of Nielsen polylogarithms, is generated by the Laurent expansion of a Gauss hypergeometric function with integer values of parameters. In general, the explicit form of this function is not uniquely determined, and the result may be presented in another form by using a different subset of harmonic polylogarithms.

2.1.2 Half-integer values of ε -independent parameters

Let us apply a similar analysis for the second basis hypergeometric function

$${}_2F_1 \left(\begin{matrix} a_1\varepsilon, a_2\varepsilon \\ \frac{1}{2} + f\varepsilon \end{matrix} \middle| z \right). \tag{2.26}$$

In this case, the differential equation has the form

$$\frac{d}{dz} \left(z \frac{d}{dz} - \frac{1}{2} + f\varepsilon \right) w(z) = \left(z \frac{d}{dz} + a_1\varepsilon \right) \left(z \frac{d}{dz} + a_2\varepsilon \right) w(z), \tag{2.27}$$

with the same boundary conditions $w(0) = 1$ and $z \frac{d}{dz} w(z)|_{z=0} = 0$. Using the ε -expanded form of the solution, and noting that eq. (2.8), and in fact, eq. (2.27) is valid at each order of the ε -expansion, we may rewrite eq. (2.27) as

$$\left[(1-z) \frac{d}{dz} - \frac{1}{2z} \right] \left(z \frac{d}{dz} \right) w_i(z) = \left[(a_1+a_2) - \frac{f}{z} \right] \left(z \frac{d}{dz} \right) w_{i-1}(z) + a_1 a_2 w_{i-2}(z). \tag{2.28}$$

Let us introduce the new variable y such that,⁹

$$y = \frac{1 - \sqrt{\frac{z}{z-1}}}{1 + \sqrt{\frac{z}{z-1}}}, \quad z = -\frac{(1-y)^2}{4y}, \quad 1-z = \frac{(1+y)^2}{4y}, \quad z \frac{d}{dz} = -\frac{1-y}{1+y} y \frac{d}{dy}, \tag{2.29}$$

and define a set of a new functions $\rho_i(y)$ so that¹⁰

$$z \frac{d}{dz} w_i(z) \equiv \left(-\frac{1-y}{1+y} y \frac{d}{dy} \right) w_i(y) = \frac{1-y}{1+y} \rho_i(y), \tag{2.30}$$

⁷We are indebted to A. Davydychev for this relation.

⁸Compare with results of [35].

⁹The form of this variable follows from the analysis performed in refs. [3, 23, 24].

¹⁰We may note that

$${}_2F_1 \left(\begin{matrix} 1 + a_1\varepsilon, 1 + a_2\varepsilon \\ \frac{3}{2} + f\varepsilon \end{matrix} \middle| z \right) = \frac{1 + 2f\varepsilon}{2z} \frac{1-y}{1+y} \sum_{k=0}^{\infty} \left[\frac{\rho_{k+2}(y)}{a_1 a_2} \right] \varepsilon^k.$$

and, as in the previous case,

$$\rho(y) = z \frac{d}{dz} w(z) = \sum_{k=0}^{\infty} \rho_k(y) \varepsilon^k . \tag{2.31}$$

In terms of the new variable y , eq. (2.28) can be written as system of two first order differential equations:

$$\begin{aligned} y \frac{d}{dy} \rho_i(y) &= (a_1 + a_2) \frac{1-y}{1+y} \rho_{i-1}(y) + 2f \left(\frac{1}{1-y} - \frac{1}{1+y} \right) \rho_{i-1}(y) + a_1 a_2 w_{i-2}(y) , \\ y \frac{d}{dy} w_k(y) &= -\rho_k(y) . \end{aligned} \tag{2.32}$$

The solution of these differential equations for functions $w_i(y)$ and $\rho_i(y)$ has the form

$$\begin{aligned} \rho_i(y) &= \int_1^y dt \left[2f \frac{1}{1-t} - 2(a_1 + a_2 - f) \frac{1}{1+t} \right] \rho_{i-1}(t) - (a_1 + a_2) [w_{i-1}(y) - w_{i-1}(1)] \\ &\quad + a_1 a_2 \int_1^y \frac{dt}{t} w_{i-2}(t) , \quad i \geq 1 , \\ w_i(y) &= - \int_1^y \frac{dt}{t} \rho_i(t) , \quad i \geq 1 . \end{aligned} \tag{2.33}$$

The point $z = 0$ transforms to the point $y = 1$ under the transformation (2.29), so that the boundary conditions are

$$\begin{aligned} w_k(1) &= 0 , \quad k \geq 1 , \\ \rho_k(1) &= 0 , \quad k \geq 0 . \end{aligned} \tag{2.34}$$

The first several coefficients of the ε -expansion can be calculated quite easily by using $w_0(y) = 1$ and $\rho_0(y) = 0$:

$$\rho_1(y) = w_1(y) = 0, \tag{2.35a}$$

$$\frac{\rho_2(y)}{a_1 a_2} = \ln(y) \equiv H(0; y) , \tag{2.35b}$$

$$\frac{w_2(y)}{a_1 a_2} = -\frac{1}{2} \ln^2(y) \equiv -H(0, 0; y) . \tag{2.35c}$$

Continuing these iterations, we may reproduce the coefficients of the ε -expansion of the Gauss hypergeometric function (2.26) presented in eq. (4.2) of ref. [25]. Since the length of the expressions obtained for the coefficient functions $\rho_3(y), \omega_3(y), \rho_4(y), \omega_4(y), \rho_5(y), \omega_5(y)$ is similar to those published in eq. (4.1) of ref. [25], we don't reproduce them here.¹¹ The higher order terms of ε -expansion are relatively lengthy and therefore will also not be presented here. Unfortunately, as in the previous case, we are unable to calculate the k -coefficient of ε -expansion without knowledge of previous ones.

From representation (2.33) we deduce the following result:

¹¹M.Y.K. thanks to M. Rogal for pointing out a mistake in eq. (4.1) of ref. [25]: In the ε^2 term, the coefficient should be “ $-2(3f - a_1 - a_2)$ ” instead of “ $-2(3f - 2a_1 - 2a_2)$ ”.

Corollary 4. *The all-order ε -expansion of function (2.26) can be written in terms of harmonic polylogarithms $H_{\vec{A}}(y)$ of variable y defined in (2.29) and multiple index \vec{A} with entries taking values 0, 1 and -1 .*

This statement follows from the representation (2.33), the values of coefficients functions $w_k(z)$, $k = 0, 1, 2$ (see eqs. (2.11), (2.17)), properties of harmonic polylogarithms, and the relation between powers of logarithms and harmonic polylogarithms. Also, Corollary 2 and Corollary 3 are valid for the hypergeometric function (2.26).

We would like to mention that, in contrast to the eq. (2.16), eq. (2.33) contains a new type of function, coming from the integral $\int f(t)dt/(1+t)$. Another difference is that the first nontrivial coefficient function, $\rho_2(y)$, is equal to $\ln(y)$, instead of $\ln(1-z)$, as it was in the previous case. It was shown in ref. [20] that terms containing the logarithmic singularities can be explicitly factorised (see eqs. (21)-(22) in ref. [20]), so that the coefficient functions, $w_k(y)$ and $\rho_k(y)$ from eq. (2.33), have the form

$$\begin{aligned}
 w_k(y) &= \sum_{j=0}^k c(\vec{s}, \vec{\sigma}, k) \ln^{k-j}(y) \left[\text{Li}_{\left(\frac{\vec{\sigma}}{\vec{s}}\right)}(y) - \text{Li}_{\left(\frac{\vec{\sigma}}{\vec{s}}\right)}(1) \right], \\
 \rho_k(y) &= \sum_{j=0}^{k-1} \tilde{c}(\vec{s}, \vec{\sigma}, k) \ln^{k-j}(y) \left[\text{Li}_{\left(\frac{\vec{\sigma}}{\vec{s}}\right)}(y) - \text{Li}_{\left(\frac{\vec{\sigma}}{\vec{s}}\right)}(1) \right],
 \end{aligned}
 \tag{2.36}$$

where $c(\vec{s}, \vec{\sigma}, k)$ and $\tilde{c}(\vec{s}, \vec{\sigma}, k)$ are numerical coefficients, \vec{s} and $\vec{\sigma}$ are multi-index, $\vec{s} = (s_1, \dots, s_n)$ and $\vec{\sigma} = (\sigma_1, \dots, \sigma_n)$, σ_k belongs to the set of the square roots of unity, $\sigma_k = \pm 1$, and $\text{Li}_{\left(\frac{\vec{\sigma}}{\vec{s}}\right)}(y)$ is a coloured multiple polylogarithm of one variable [15–17], defined as

$$\text{Li}_{\left(\frac{\sigma_1, \sigma_2, \dots, \sigma_k}{s_1, s_2, \dots, s_n}\right)}(z) = \sum_{m_1 > m_2 > \dots > m_n > 0} z^{m_1} \frac{\sigma_1^{m_1} \dots \sigma_n^{m_n}}{m_1^{s_1} m_2^{s_2} \dots m_n^{s_n}}.
 \tag{2.37}$$

It has an iterated integral representation w.r.t. three differential forms,

$$\begin{aligned}
 \omega_0 &= \frac{dy}{y}, \quad \sigma = 0, \\
 \omega_\sigma &= \frac{\sigma dy}{1 - \sigma y}, \quad \sigma = \pm 1,
 \end{aligned}
 \tag{2.38}$$

so that,

$$\text{Li}_{\left(\frac{\sigma_1, \sigma_2, \dots, \sigma_k}{s_1, s_2, \dots, s_k}\right)}(y) = \int_0^1 \omega_0^{s_1-1} \omega_{\sigma_1} \omega_0^{s_2-1} \omega_{\sigma_1 \sigma_2} \dots \omega_0^{s_k-1} \omega_{\sigma_1 \sigma_2 \dots \sigma_k}, \quad \sigma_k^2 = 1.
 \tag{2.39}$$

The values of coloured polylogarithms of unit argument were studied in refs. [36, 37].

2.2 Zero-values of the ε -dependent part of upper parameters

In the case when one of the upper parameter of the Gauss hypergeometric function is a positive integer, the result of the reduction has the simpler form (compare with eq. (2.1)):

$$P(b, c, z) {}_2F_1(I_1, b + I_2; c + I_3; z) = Q_1(b, c, z) {}_2F_1(1, b; c; z) + Q_2(b, c, z),
 \tag{2.40}$$

where b, c , are any fixed numbers, P, Q_1, Q_2 are polynomial in parameters b, c and argument z , and I_1, I_2, I_3 are any integers.¹² In this case, it is enough to consider the following two basis functions: ${}_2F_1(1, 1+a\varepsilon; 2+c\varepsilon; z)$ and ${}_2F_1(1, 1+a\varepsilon; \frac{3}{2}+f\varepsilon; z)$. The ε -expansion of this function can be derived from the proper solution given by eq. (2.17) or eq. (2.35), using the relations

$${}_2F_1\left(\begin{matrix} 1, 1+a_2\varepsilon \\ 2+f\varepsilon \end{matrix} \middle| z\right) = \lim_{a_1 \rightarrow 0} \frac{1+c\varepsilon}{a_1 a_2 \varepsilon^2} \frac{d}{dz} {}_2F_1\left(\begin{matrix} a_1\varepsilon, a_2\varepsilon \\ 1+c\varepsilon \end{matrix} \middle| z\right) = \frac{1+c\varepsilon}{z} \sum_{k=0}^{\infty} \left[\frac{\rho_{k+2}(z)}{a_1 a_2} \middle|_{a_1=0} \right] \varepsilon^k \tag{2.41}$$

and

$${}_2F_1\left(\begin{matrix} 1, 1+a_2\varepsilon \\ \frac{3}{2}+f\varepsilon \end{matrix} \middle| z\right) = \lim_{a_1 \rightarrow 0} \frac{1+2f\varepsilon}{2a_1 a_2 \varepsilon^2} \frac{d}{dz} {}_2F_1\left(\begin{matrix} a_1\varepsilon, a_2\varepsilon \\ \frac{1}{2}+f\varepsilon \end{matrix} \middle| z\right) = \frac{1+2f\varepsilon}{2z} \sum_{k=0}^{\infty} \left[\frac{\rho_{k+2}(y)}{a_1 a_2} \middle|_{a_1=0} \right] \varepsilon^k, \tag{2.42}$$

where we have used the differential relation

$$\frac{d}{dz} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| z\right) = \frac{ab}{c} {}_2F_1\left(\begin{matrix} 1+a, 1+b \\ 1+c \end{matrix} \middle| z\right),$$

and the brackets mean that in the proper solution, we can put $a_1 = 0$. The functions ρ_k are given by eq. (2.17) and eq. (2.35), correspondingly. Due to Corollary 3, the limit $a_1 \rightarrow 0$ must exist.

The case when both upper parameters are integers may be handled in a similar manner. Theorem 1 is thus proved. ■

3. Some particular cases

3.1 The generalized log-sine functions and their generalization

For the case $0 \leq z \leq 1$ the variable y defined in (2.29) belongs to a complex unit circle, $y = \exp(i\theta)$. In this case, the harmonic polylogarithms can be split into real and imaginary parts (see the discussion in appendix A of ref. [24]), as in the case of classical polylogarithms. [33] Let us introduce the trigonometric parametrization $z = \sin^2 \frac{\theta}{2}$. In this case, the solution of the proper differential equations (2.16) and (2.33) can be written in the form

$$\begin{aligned} \rho_i(\theta) &= (a_1+a_2-c) \int_0^\theta d\phi \frac{\sin \frac{\phi}{2}}{\cos \frac{\phi}{2}} \rho_{i-1}(\phi) + a_1 a_2 \int_0^\theta d\phi \frac{\sin \frac{\phi}{2}}{\cos \frac{\phi}{2}} w_{i-2}(\phi) - c w_{i-1}(\theta), \quad i \geq 1, \\ w_i(\theta) &= \int_0^\theta d\phi \frac{\cos \frac{\phi}{2}}{\sin \frac{\phi}{2}} \rho_i(\phi), \quad i \geq 1, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \rho_i(\theta) &= (a_1+a_2-f) \int_0^\theta d\phi \frac{\sin \frac{\phi}{2}}{\cos \frac{\phi}{2}} \rho_{i-1}(\phi) - f \int_0^\theta d\phi \frac{\cos \frac{\phi}{2}}{\sin \frac{\phi}{2}} \rho_{i-1}(\phi) + a_1 a_2 \int_0^\theta d\phi w_{i-2}(\phi), \\ w_i(\theta) &= \int_0^\theta d\phi \rho_i(\phi), \quad i \geq 1, \end{aligned} \tag{3.2}$$

¹²The proper algebraic relations for the reduction are given in ref. [25].

respectively. In the first case, the solutions of the system of equations (3.1) are harmonic polylogarithms with argument equal to $\sin^2 \frac{\theta}{2}$. In the second case, the result contains the generalized log-sine functions [33, 38–40] and some of their generalizations studied in ref. [41] (see also ref. [42]). For illustration, we will present a first several terms of the ε -expansion¹³ (see the proper relations, table I of appendix C in ref. [23]):

$$\begin{aligned}
 & {}_2F_1 \left(\begin{matrix} 1 + a_1\varepsilon, 1 + a_2\varepsilon \\ \frac{3}{2} + f\varepsilon \end{matrix} \middle| \sin^2 \frac{\theta}{2} \right) = \frac{(1 + 2f\varepsilon)}{\sin \theta} \\
 & \times \left(\theta + 2\varepsilon \left\{ \gamma_f (\text{Ls}_2(\pi - \theta) - \theta L_\theta) - f (\text{Ls}_2(\theta) + \theta l_\theta) \right\} \right. \\
 & + \varepsilon^2 \left\{ 2f\gamma_{2f} \text{Ls}_3(\theta) + 2\gamma_f\gamma_{2f} \text{Ls}_3(\pi - \theta) - f\gamma_f \text{Ls}_3(2\theta) \right. \\
 & \quad + 4f\gamma_f [\text{Ls}_2(\theta) L_\theta - \text{Ls}_2(\pi - \theta) l_\theta + \theta L_\theta l_\theta] + 4f^2 \text{Ls}_2(\theta) l_\theta - 4\gamma_f^2 \text{Ls}_2(\pi - \theta) L_\theta \\
 & \quad \left. + 2f^2 \theta l_\theta^2 + 2\gamma_f^2 \theta L_\theta^2 + \frac{1}{6} a_1 a_2 \theta^3 + \gamma_f \gamma_{2f} \pi \zeta_2 \right\} \\
 & + \varepsilon^3 \left\{ \frac{4}{3} \gamma_{2f} [(a_1 + a_2) \gamma_f \text{Ls}_4(\pi - \theta) + f^2 \text{Ls}_4(\theta) - 3f\gamma_f \text{Lsc}_{2,3}(\theta)] - \frac{2}{3} f^2 \gamma_f \text{Ls}_4(2\theta) \right. \\
 & \quad + 4a_1 a_2 [2f \text{Cl}_4(\theta) - 2\gamma_f \text{Cl}_4(\pi - \theta) - f \text{Cl}_3(\theta) \theta - \gamma_f \text{Cl}_3(\pi - \theta) \theta] \\
 & \quad + 2 [f l_\theta + \gamma_f L_\theta] [f \gamma_f \text{Ls}_3(2\theta) - 2\gamma_f \gamma_{2f} \text{Ls}_3(\pi - \theta) - 2f \gamma_{2f} \text{Ls}_3(\theta)] \\
 & \quad + 2 [f l_\theta + \gamma_f L_\theta]^2 [2\gamma_{2f} \text{Ls}_2(\pi - \theta) - f \text{Ls}_2(2\theta)] + a_1 a_2 \gamma_{2f} \theta^2 \text{Ls}_2(\pi - \theta) \\
 & \quad - \frac{1}{2} a_1 a_2 f \text{Ls}_2(2\theta) \theta^2 - \frac{1}{3} a_1 a_2 \theta^3 [f l_\theta + \gamma_f L_\theta] - \frac{4}{3} \theta [f l_\theta + \gamma_f L_\theta]^3 - 2\gamma_f \gamma_{2f} \pi \zeta_2 [f l_\theta + \gamma_f L_\theta] \\
 & \quad \left. + a_1 a_2 (3a_1 + 3a_2 - 7f) \theta \zeta_3 - 2(a_1 + a_2) \gamma_f \gamma_{2f} \pi \zeta_3 \right\} + \mathcal{O}(\varepsilon^4) \tag{3.3}
 \end{aligned}$$

and

$$\begin{aligned}
 & {}_2F_1 \left(\begin{matrix} a_1\varepsilon, a_2\varepsilon \\ \frac{1}{2} + f\varepsilon \end{matrix} \middle| \sin^2 \frac{\theta}{2} \right) = 1 + a_1 a_2 \varepsilon^2 \left(\frac{1}{2} \theta^2 \right. \\
 & \quad \left. + \varepsilon \left\{ 2f \text{Ls}_2(\theta) \theta - 2\gamma_f \text{Ls}_2(\pi - \theta) \theta + 4\gamma_f \text{Cl}_3(\pi - \theta) + 4f \text{Cl}_3(\theta) + (3a_1 + 3a_2 - 7f) \zeta_3 \right\} \right. \\
 & \quad \left. + \varepsilon^2 \left\{ 2\gamma_f \gamma_{2f} \text{Ls}_3(\pi - \theta) \theta - f \gamma_f \text{Ls}_3(2\theta) \theta + 2f \gamma_{2f} \text{Ls}_3(\theta) \theta + \frac{1}{24} a_1 a_2 \theta^4 \right. \right. \\
 & \quad \left. \left. - 2 [f \text{Ls}_2(\theta) - \gamma_f \text{Ls}_2(\pi - \theta)]^2 + \gamma_f \gamma_{2f} \theta \pi \zeta_2 \right\} + \mathcal{O}(\varepsilon^3) \right), \tag{3.4}
 \end{aligned}$$

where

$$L_\theta = \ln \left(2 \cos \frac{\theta}{2} \right), \quad l_\theta = \ln \left(2 \sin \frac{\theta}{2} \right),$$

¹³The FORM representation of these expressions can be extracted from [48].

the generalized log-sine function is defined as

$$\text{Ls}_j^{(k)}(\theta) = - \int_0^\theta d\phi \phi^k \ln^{j-k-1} \left| 2 \sin \frac{\phi}{2} \right|, \quad \text{Ls}_j(\theta) = \text{Ls}_j^{(0)}(\theta), \quad (3.5)$$

and we use the notation $\text{Lsc}_{2,3}(\theta)$ for the special combination (see eq. (2.18) in ref. [23])

$$\begin{aligned} \text{Lsc}_{2,3}(\theta) &= \frac{1}{12} \text{Ls}_4(2\theta) - \frac{1}{3} \text{Ls}_4(\theta) + 2 \text{Ti}_4\left(\tan \frac{\theta}{2}\right) - 2 \ln\left(\tan \frac{\theta}{2}\right) \text{Ti}_3\left(\tan \frac{\theta}{2}\right) \\ &\quad + \ln^2\left(\tan \frac{\theta}{2}\right) \text{Ti}_2\left(\tan \frac{\theta}{2}\right) - \frac{1}{6} \theta \ln^3\left(\tan \frac{\theta}{2}\right), \end{aligned} \quad (3.6)$$

where the functions $\text{Ti}_N(z)$ are defined as [33]

$$\text{Ti}_N(z) = \text{Im}[\text{Li}_N(iz)] = \frac{1}{2i} [\text{Li}_N(iz) - \text{Li}_N(-iz)], \quad \text{Ti}_N(z) = \int_0^z \frac{dx}{x} \text{Ti}_{N-1}(x). \quad (3.7)$$

These functions receive special interest in physics through their role in the so-called “single-scale” diagrams, which depend only on one massive scale parameter. The massless propagator-type diagrams, bubble-type diagrams and propagator-type diagrams on mass shell all belong to this class. In particular, the single-scale diagrams with two massive particle cuts correspond to hypergeometric functions with value of argument equal to $z = 1/4$. [43, 22, 3] In this case, the value of the conformal variable y is equal to the primitive “sixth root of unity”, $y = \exp(i\pi/3)$. In contrast to the case in multiple polylogarithms (2.6) of the primitive sixth root of unity studied in ref. [44] and the more complicated case in coloured polylogarithms of the sixth root of unity studied by Broadhurst in ref. [16], the physically interesting case corresponds to coloured polylogarithms of square root (2.39) (harmonic polylogarithms) with argument equal to primitive sixth root of unity. In this case, some new transcendental constants, in addition to studied in ref. [44] will be generated. The set of independent constants up to weight 5 was constructed in refs. [38, 3, 41].

3.2 Special cases: all-order ε -expansion in terms of Nielsen polylogarithms

One advantage of a trigonometric representation used in the previous section is the theorem proved in ref. [3] (see also ref. [40]), that any generalized log-sine function (3.5) is expressible in term of Nielsen polylogarithms [34] only. Using this theorem, it was shown in ref. [3, 23] that for the Gauss hypergeometric function

$${}_2F_1\left(\begin{matrix} 1, 1+a\varepsilon \\ \frac{3}{2}+b\varepsilon \end{matrix} \middle| \sin^2 \frac{\theta}{2}\right), \quad (3.8)$$

the Laurent expansion is expressible in terms of only Nielsen polylogarithms in the three cases (i) $b = 0$, (ii) $b = a$, (iii) $a = 2b$. Using the reduction algorithm [25], we can claim that the Laurent expansions of the following functions are also expressible in terms of Nielsen polylogarithms only:

$${}_2F_1\left(\begin{matrix} I_1, I_2+\varepsilon \\ \frac{1}{2}+I_3 \end{matrix} \middle| \sin^2 \frac{\theta}{2}\right), \quad {}_2F_1\left(\begin{matrix} I_1, I_2+\varepsilon \\ \frac{1}{2}+I_3+\varepsilon \end{matrix} \middle| \sin^2 \frac{\theta}{2}\right), \quad {}_2F_1\left(\begin{matrix} I_1, I_2+\varepsilon \\ \frac{1}{2}+I_3+\frac{1}{2}\varepsilon \end{matrix} \middle| \sin^2 \frac{\theta}{2}\right), \quad (3.9)$$

a	I_1	I_1	I_1	I_1	I_1	I_1	$I_1 + \varepsilon$	$I_1 + \varepsilon$	$I_1 + 2\varepsilon$
b	$\frac{1}{2} + I_2$	$\frac{1}{2} + I_2 + \varepsilon$	$\frac{1}{2} + I_2 - \varepsilon$	$\frac{1}{2} + I_2 + \varepsilon$	$\frac{1}{2} + I_2$	$\frac{1}{2} + I_2 + \varepsilon$	$\frac{1}{2} + I_2$	$\frac{1}{2} + I_2 + \varepsilon$	$\frac{1}{2} + I_2 + \varepsilon$
c	$\frac{1}{2} + I_3 + \varepsilon$	$\frac{1}{2} + I_3$	$\frac{1}{2} + I_3 + \varepsilon$	$I_3 + \varepsilon$	$I_3 + \varepsilon$	$I_3 + 2\varepsilon$	$I_1 + 1 + \varepsilon$	$I_1 + 1 + \varepsilon$	$I_1 + 1 + 2\varepsilon$

Table 1:

where I_1 , I_2 and I_3 are integers.

It is interesting to analyze this solution from the point of view of eq. (3.2). Due to fact that $a_1 = 0$, the last term in eq. (3.2) is identically equal to zero. In case (i), only the first term survives, with integration kernel having the form $d \ln(\cos \frac{\phi}{2})$. In case (ii), only the second term survives, and the integration kernel has the form $d \ln(\sin \frac{\phi}{2})$. In case (iii), the first and second terms can be reduced to the second case of a double argument. The statement about expressibility of *inverse binomial sums* in terms of log-sine function, proved in ref. [23] (see also ref. [24]) applies to all three of these cases.

We can extend the class of Gauss functions whose ε -expansions are expressible in terms of only Nielsen polylogarithms by using algebraic relations¹⁴ between of the fractional-linear arguments (see section 3 in ref. [25]). The cases which may be expressed in this manner are summarized in table 1, where a, b, c are parameters of the Gauss hypergeometric functions ${}_2F_1(a, b; c; z)$ and I_1, I_2 and I_3 are integer:

The results of this section can be formulated as follows:

Proposition 1. *All cases of Gauss hypergeometric functions with half-integer values of parameters for which the all-order ε -expansion is expressible in terms of only Nielsen polylogarithms are described in eq. (3.9) or the parameters shown in table I.*

4. Conclusions

The main result of this paper is the proof of Theorem 1, as stated also in the abstract. The proof includes two steps: (i) the algebraic reduction of Gauss hypergeometric functions of the type in Theorem 1 to basic functions and (ii) the iterative algorithms for calculating the analytical coefficients of the ε -expansion of basic hypergeometric functions.

In implementing step (i), the algebraic relations between basis functions with half-integer values of parameters reduce all of the cases to the one basic function of type (2.2a) and its first derivative (see details in ref. [25]). In step (ii), the algorithm is constructed for integer values of parameters in eq. (2.16) and for basis Gauss hypergeometric functions with half-integer values of parameters in eq. (2.33). This allows us to calculate the coefficients directly, without reference to multiple sums.

It is interesting to note that the Laurent expansions of the Gauss hypergeometric functions with integer values of parameters are expressible in terms of multiple polylogarithms of one variable (see eq. (2.6)) or the Remiddi-Vermaseren harmonic polylogarithms with multiple index including only values 0 and 1. The argument of the resulting functions coincides with the original variable of the hypergeometric function. For Gauss hypergeometric

¹⁴This can also be derived via the integral representation.

functions with half-integer values of parameters, the coefficients of the ε -expansion produce the full set of harmonic polylogarithms, or coloured multiple polylogarithms of one variable (see eq. (2.37)). These functions depend on a new variable, related to the original variable by conformal transformation (see ref. [25]).

For special values of the argument of the hypergeometric function, $z < 1$, the coloured multiple polylogarithms of one variable may be split into real and imaginary parts. This case has been discussed in section 3.1. It was shown that the physically interesting case, representing single-scale diagrams with two massive particle cuts, corresponds to coloured polylogarithms (2.39) with argument equal to a primitive “sixth root of unity”, $y = \exp\left(i\frac{\pi}{3}\right)$. This gives an explanation of the proper “basis of transcendental constants” constructed in refs. [38] and [3], and its difference from the proper basis of David Broadhurst [16].

In the section 3.2, the subset of Gauss hypergeometric function is analyzed, showing that the all-order ε -expansion is expressible in terms of Nielsen polylogarithms only. In particular, we have formulated the proposition that the only Gauss hypergeometric functions with half-integer values of parameters for which the all-order ε -expansion is expressible in terms of Nielsen polylogarithms only belong to one of the functions described in (3.9) or in table I.

In appendix A, we discuss the construction of the all-order Laurent expansion of the Gauss hypergeometric function (2.26) around $z = 1$.

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A. The Laurent expansion of Gauss hypergeometric functions with half-integer values of parameters around $z = 1$

The identities between harmonic polylogarithms (2.6) under the action of the group of fractional-linear transformation of the argument,

$$z \rightarrow 1 - z, \quad \frac{1}{z}, \quad z^2,$$

was considered in ref. [20] (see also ref. [46]). It was shown [20] that the full set of Remiddi-Vermaseren functions is invariant with respect to transformations

$$z \rightarrow \frac{1}{z}, \quad \frac{1-z}{1+z}.$$

In this appendix, we wish to show that the coefficient functions $\omega_k(y)$ and $\rho_k(y)$ entering in the Laurent expansion of the hypergeometric function (2.26) satisfy to some identities with respect to the argument transformation $z \rightarrow 1 - z$. The derivation is trivial if we recall that the all-order Laurent expansion of the hypergeometric function (2.26) can be written in terms of coloured polylogarithms of argument y . Under the transformation $z \rightarrow 1 - z$, the conformal variable simply changes its sign, $y \mapsto -y$. The full set of Remiddi-Vermaseren functions is invariant (up to the addition of a constant imaginary part) with respect to changing of the sign of this variable. Consequently, the coefficient functions $\omega_k(y)$ and $\rho_k(y)$ should be related to $\omega_k(-y)$ and $\rho_k(-y)$.

Let us present the explicit relations. Using the Kummer's relations between hypergeometric functions of arguments z and $1 - z$, we obtain

$$\begin{aligned}
 & {}_2F_1 \left(\begin{matrix} a_1\varepsilon, a_2\varepsilon \\ \frac{1}{2} + f\varepsilon \end{matrix} \middle| 1 - z \right) \\
 &= - \frac{z^{1/2 + (f - a_1 - a_2)\varepsilon} \Gamma(\frac{1}{2} + f\varepsilon) \Gamma(\frac{1}{2} + (a_1 + a_2 - f)\varepsilon)}{(1 - z)^{-1/2 + f\varepsilon} \Gamma(1 + a_1\varepsilon) \Gamma(1 + a_2\varepsilon)} \frac{d}{dz} {}_2F_1 \left(\begin{matrix} -a_1\varepsilon, -a_2\varepsilon \\ \frac{1}{2} + (f - a_1 - a_2)\varepsilon \end{matrix} \middle| z \right) \\
 &+ \frac{\Gamma(\frac{1}{2} + f\varepsilon) \Gamma(\frac{1}{2} + (f - a_1 - a_2)\varepsilon)}{\Gamma(\frac{1}{2} + (f - a_1)\varepsilon) \Gamma(\frac{1}{2} + (f - a_2)\varepsilon)} {}_2F_1 \left(\begin{matrix} a_1\varepsilon, a_2\varepsilon \\ \frac{1}{2} + (a_1 + a_2 - f)\varepsilon \end{matrix} \middle| z \right). \tag{A.1}
 \end{aligned}$$

The all-order ε -expansion for the hypergeometric functions entering in r.h.s. of this relation is constructed in section 2.1.2. Let us apply the same technique for constructing the Laurent expansion of the hypergeometric function on the l.h.s. In accordance with standard procedure (see ref. [28]), let us introduce a new variable, $Z = 1 - z$, so that the differential equation around $Z = 0$ has the form

$$\frac{d}{dZ} \left(Z \frac{d}{dZ} - \frac{1}{2} + (a_1 + a_2 - f)\varepsilon \right) w(Z) = \left(Z \frac{d}{dZ} + a_1\varepsilon \right) \left(Z \frac{d}{dZ} + a_2\varepsilon \right) w(Z), \tag{A.2}$$

This equation is equivalent to eq. (2.27) with the proper change of variable and one of the parameters,

$$(z, f) \longleftrightarrow (Z, a_1 + a_2 - f),$$

so that we can use the results of section 2.1.2 with the proper change of notations. In particular, the solutions of the differential equations for the functions $\rho_i(Z)$ and $w_i(Z)$ have the form

$$\begin{aligned}
 \rho_i(Y) &= \int_1^Y dt \left[2(a_1 + a_2 - f) \frac{1}{1 - t} - 2f \frac{1}{1 + t} \right] \rho_{i-1}(t) - (a_1 + a_2) [w_{i-1}(Y) - w_{i-1}(1)] \\
 &+ a_1 a_2 \int_1^Y \frac{dt}{t} w_{i-2}(t), \quad i \geq 1, \\
 w_i(Y) &= - \int_1^Y \frac{dt}{t} \rho_i(t), \quad i \geq 1. \tag{A.3}
 \end{aligned}$$

where new variable Y is defined as

$$Y = \frac{1 - \sqrt{\frac{Z}{Z-1}}}{1 + \sqrt{\frac{Z}{Z-1}}} \equiv -y, \tag{A.4}$$

and y is defined by eq. (2.29). In this way, both parts of relation (A.1) are expressible in terms of coloured polylogarithms depending on the arguments y (r.h.s.) and $-y$ (l.h.s.).

We expect that relations following from eq. (A.1) may be useful in obtaining some dual relations for coloured polylogarithms (see ref. [16, 17, 44]), and in the obtaining algebraic relations between coloured polylogarithms of the primitive “sixth root of unity”, as in case of multiple zeta-values [47]. At the present moment, we are not ready to discuss these relations.

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